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Unidirectional drift of bistable front under asymmetrically oscillating zero-mean force

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The unidirectional drift of bistable fronts (BFs) that separate two stable uniform states of a bistable system under the action of an asymmetrically oscillating zero-mean force (driver) is considered within the "pseudolinear" (piecewise-linear) model of the system. The particular case of the *symmetrical* (symmetrically shaped) rate functions is studied. To perform a rigorous analytic treatment for arbitrary strengths of the driving force we assume that the applied ac force is quasistatically slow. Both cases of the initially static and the initially propagating BFs are examined; various types of the "unforced" dc motion are found. We show that the unforced transport of BF takes place in any case of the asymmetric driver, whether Maxwellian construction of the rate function was balanced or not. In particular, progressive (accelerated) dc drift of the initially static BFs occurred. In contrast, both progressive and regressive (decelerated) types of unforced dc drift of the initially propagating BFs take place. Moreover, reversal of the directed motion of the initially propagating BF occurred, if the deviation of Maxwellian construction from the strictly balanced situation was relatively small; by tuning the strength of the driving force the dc drift of BF exhibits the reversal. The symmetry properties of the biharmonic driver are discussed. The biharmonic ac force consisting of a superposition of the fundamental mode and its even (odd) superharmonics is an asymmetrically (symmetrically) oscillating one. The reversal type of the unforced dc drift occurred only in the case of the even superharmonic "mixing," when the superharmonic mode of the biharmonic driver was even.

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I. INTRODUCTION

The ratchet effect, a systematic direct motion of the free particles, stimulated by zero-mean forces (driver), arises in a large class of nonlinear dissipative systems out of thermal equilibrium. Both versions of the stochastic (noisy driver) and deterministic (regular driver) ratchets are possible and have been discussed at length in Refs. [1]. A crucial feature of the ratchet "device" is the spatial asymmetry of the substrate potential. The ordinary ratchet (Brownian motor, noisy and deterministic one) works as a particle separator in a spatially periodic system that is characterized by asymmetrically shaped potential; the unidirectional transport takes place if the "active" particles experience the spatial asymmetry of the substrate potential. The considered ratchet effect takes place too even in the case of the externally induced asymmetry; the "unforced" drift generated by "kinematical" asymmetry of the substrate potential has been recently discussed in Ref. [2].

A topic currently receiving much attention is the unidirectional transport of the solitary self-ordered "states" (front-structures) in the nonlinear, spatially uniform systems under the action of the deterministic or noisy zero-mean forces. Two different mechanisms underlying the unforced motion discussed, namely, the parametrically (externally) stimulated and directly (internally) induced transport of the self-localized fronts, both coherent and dissipative ones, have been studied analytically and by numerical simulations too (see, e.g., the papers in [3–8]). Recent studies carried out with the sine-Gordon, cubic polynomial and pseudolinear (piecewise-linear) models of the underdamped, moderately damped and extremely overdamped systems being under the action of the periodic zero-mean force showed that the unforced transport in "front-ratchets" is sensitive to the temporal symmetry of the driving force $f(t)$ (e.g., see Refs. [4–8]).

In the present paper the unforced transport of the bistable fronts (BFs) that separate two stable uniform states of the bistable dissipative (extremely overdamped) system of the reaction-diffusion type is considered. We extend our previous study of the deterministic "front-ratchet" (DFR) presented in [7,8] by consideration of the "asymmetrically" driven BFs under the action of the asymmetrically oscillating zero-mean force. The response of BF to the applied ac force is described by the following equation,

$$u_t - u_{zz} - c(t)u_z + R(u) = f(t), \tag{1}$$

where the function $u(z,t)$ denotes the steplike field of the front propagating at the moment velocity $c(t)$, $z=x-ct$ is the traveling coordinate, and the rate function $R(u)$, which characterizes the rate of the transient processes in the system, has three zeros at $u=u_1,u_2,u_3$ (say, $u_1<u_2<u_3$). In the considered case of the bistable system one has that $R'(u_{1,3})>0$, and $R'(u_2)<0$, where the prime denotes the derivative. The front solution of the free ($f\equiv 0$) BF $u_0(z)$ joins two fixed points (stable uniform states) u_1 and u_3, namely; the following relations hold: $u_0(z\to -\infty)\to u_1$, and $u_0(z\to\infty)\to u_3$. Recent studies carried out within the cubic polynomial and pseudolinear models of the bistable system showed that the characteristic features of the unforced transport of BFs were sensitive to the symmetry properties of both the rate function and the driving force [7,8]. More specifically, the peculiarities of $v-f_a$ characteristics, that describe the dependence of the drift

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(mean) velocity v of BF versus the amplitude f_a of the driving force, depend on the shape of both functions, $R(u)$ and $f(t)$. The following combinations of the symmetry in the considered DFR have been identified: (I) $\{R_S\}-f_S$, (II) $\{R_A\}-f_S$, (III) $\{R_S\}-f_A$, and (IV) $\{R_A\}-f_A$, where the labels S and A stand for the symmetrically and nonsymmetrically shaped functions, respectively. Similarly as in the case of the ordinary ratchets, zero-mean force $f_S(t)$ is treated as symmetric (contrasymmetric) if the equality $f_S(t+T/2)=-f_S(t)$ holds, where by T we denote the period of the ac force. Differently, zero-mean force $f_A(t)$ is asymmetric if the relation $f_A(t+T/2)\neq -f_A(t)$ holds. Quite similarly, the *symmetrical* and *asymmetrical* rate functions defined by the relations $\{R_S\}:=R_S^0(u)+C$ and $\{R_A\}:=R_A^0(u)+C$ satisfy the relations $R_S^0(u_2-\Delta u)=-R_S^0(u_2+\Delta u)$ and $R_A^0(u_2-\Delta u)\neq -R_A^0(u_2+\Delta u)$, where the quantity Δu denotes the free variable, C is an arbitrary constant, and the subindex "0" means that Maxwellian construction of the rate function $R^0(u)$ was strictly balanced. In the other words, the symmetry of the associated potential, $W'(u):=-R(u)$, is rather different in either case of the *symmetrical* or *asymmetrical* rate function. Namely, the following relations hold: $W_S^0(u_2-\Delta u)=W_S^0(u_2+\Delta u)$ and $W_A^0(u_2-\Delta u)\neq W_A^0(u_2+\Delta u)$, where, as previously, the labels S and A indicate the symmetry. Clearly, the family (an infinite set) of the *similarly shaped* rate functions $\{R(u)\}$, either symmetrical or asymmetrical, is generated, by tuning the free constant C. Both the free front solutions and the propagation velocities of BF that are related to the similarly shaped rate functions $\{R(u)\}$ depend on C. In particular, the propagation velocity c_0 increases with the increasing C. Finally we notice that both the cubic polynomial and the sine-type rate functions, frequently used in considerations of the bistable fronts, are symmetrical (contrasymmetrically shaped). On the other hand, the temporal symmetry of the biharmonic ac force, frequently used in theoretical studies of the ordinary and soliton "ratchets," is flexible. Using the combination (superposition) of the odd/even harmonics one gets either symmetrically or asymmetrically oscillating zero-mean force $f(t)$.

The characteristic features of the directed drift of BFs being under the action of the symmetrically oscillating zero-mean force, both versions of DFR, above labeled by (I) and (II), have already been discussed in [7,8]. In the present paper we shall deal with case (III); the unforced drift generated by the symmetrically shaped rate functions $\{R_S\}$ and the asymmetrically oscillating zero-mean force f_A is considered. We notice that the particular, very special case of the "asymmetrically" driven BFs, namely, the dc drift of the initially static BF being under action of the biharmonic ac force consisting of the superposition of the fundamental mode and its second harmonic was recently discussed in Ref. [8].

For the sake of brevity, in what follows we shall use the abbreviation: the bistable front under the action of the symmetrically (asymmetrically) oscillating zero-mean force will be referred to as "symmetrically" (asymmetrically) driven BF.

The paper is organized as follows. In Sec. II we discuss the model and approximations. Section III deals with the speed equation of the quasistationary driven fronts. The symmetry properties of the biharmonic driver are discussed in Sec. IV. Section V deals with the unforced transport of the asymmetrically driven BFs. Both cases of the initially static and initially propagating fronts are considered. Finally, we summarize the main conclusions.

II. MODEL AND APPROXIMATIONS

The analytic treatment of the ac driven BFs requires the use of approximate approaches. The perturbative techniques that are valid in the case of the weak driving force are of limited use. Some interesting peculiarities of the unforced transport of BFs may occur at relatively high strengths of the driving force, as shown in Ref. [8]. Our principal goal is to present the main outlines of the unforced drift generated by the asymmetrically oscillating force of zero mean, in a wide interval of the amplitudes f_a of the driving force. Similarly as in [7,8], to describe the dc drift generated by the driving forces of the arbitrary strengths, we shall use an adiabatic (quasistatic) approximation. We assume that the driving force is slow, if compared to characteristic relaxation time in the system. A rough criterion of the approximation used was presented in [7,8]. It reads $T_f \gg \tau$, where the parameter τ indicates the characteristic relaxation time in the system, $\tau=\max\{R'^{-1}(u_1),R'^{-1}(u_3)\}$, and the quantity $T_f=\min\{T_n\}$ denotes the period of the rapidly oscillating mode of the ac force. Clearly, the periodic force $f(t)$ may be presented as a superposition of the harmonic modes. The adiabatic approximation is exact in the limit $T_f\to\infty$. The considered approximation was recently applied to consideration of the self-ordered states (pulled fronts, spiral waves) in the quasistationary perturbed system of the reaction-diffusion type [9]. The response of the driven BF to the quasistatic ac force is a rapid, almost instantaneous one, in the time scale of the period T_f. Thus we drop the time derivative in Eq. (1). The governing equation reads

$$u_{zz}+c(t)u_z-R_F[u;f(t)]=0,\quad R_F=R(u)-f(t), \tag{2}$$

where by $R_F(u)$ we denote the modified rate function. Similarly as in [8], the "pseudolinear" model of the bistable system is used, namely, the rate function $R(u)$ is approximated by the linear pieces. The considered rate function has a special "status:" it is flexible and analytically tractable. All above discussed versions of "front ratchets" are analytically tractable with the considered rate function, described by the following expression (see Fig. 1):

$$R(u)=\begin{cases}\alpha_1(u-u_1), & u<u_M,\\ -\alpha_2(u-u_2), & u_M<u<u_m,\\ \alpha_3(u-u_3), & u>u_m.\end{cases} \tag{3}$$

Here the free parameters are defined as follows: $u_1<u_M<u_2<u_m<u_3$ and $\alpha_i>0$, $(i=1,2,3)$. The extremes of the considered rate function, $R_M\equiv R(u_M)$ and $R_m\equiv R(u_m)$, are given by the relation $R_{M,m}=\alpha_2(u_2-u_{M,m})$. The front solutions of the pseudolinear model have been presented in Refs. [8,10].

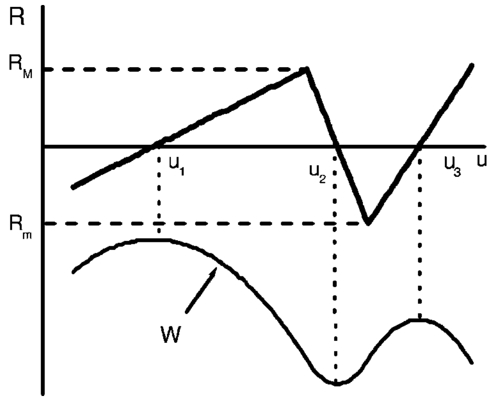

FIG. 1. The pseudolinear rate function and its associated potential, $W'(u) \coloneqq -R(u)$.

III. SPEED EQUATION OF THE ac DRIVEN FRONT

The main subject of the present consideration is the propagation velocity of the ac driven BF. Regretfully, in the considered case of the pseudolinear rate function the explicit expressions of the front solutions of BF in $u_z - u$ plane (phase trajectories) are lacking, different from the cases of the cubic-polynomial and sine-Gordon models. As a consequence, the moment velocity $c(t)$ of the quasistatically driven BF is not expressible as the function of the free parameters of the rate function and the driving force $f(t)$. More specifically, the moment velocity $c(t)$ is described by the "speed equation" being derived by the direct solution of the governing equation (2) used in conjunction with the appropriate boundary and matching conditions. The required speed equation reads (see Ref. [8])

$$\frac{Sn(s)}{\exp[-\varphi(s)]\sin\Phi(s)} = \frac{h_R - (1+h_R)f^*}{1+(1+h_R)f^*}, \tag{4a}$$

where the quantity $s(t) = c(t)/c_P$ denotes the moment velocity, scaled in the units of the "marginal" speed of the "pushed" front, $c_P = 2\sqrt{\alpha_2}$, the function $f^*(t) \coloneqq f(t)/\Delta R$ stands for the driving force, scaled in the units of the height of the rate function $\Delta R = R_M - R_m$, and the rest parameter $h_R \coloneqq -R_M/R_m$ indicates the ratio of the extreme values of the rate function. The unknown functions in Eq. (4a) are described as follows:

$$\varphi(s) = \frac{s\Phi(s)}{Q_2(s)}, \quad \Phi(s) = \begin{cases} \arctan Tg(s), & Tg(s) > 0, \\ \pi - \arctan[-Tg(s)], & Tg(s) < 0, \end{cases}$$
$$Sn(s) = F_{Sn}/F_V, \quad Tg(s) = F_{Sn}/F_{Cn}, \tag{4b}$$

and the auxiliary functions are given by the expressions

$$F_{Sn} = Q_2(s)[\delta_1 K_1(s) - \delta_3 K_3(s)],$$
$$F_{Cn} = -[Q_2^2(s) + G_1(s)G_3(s)], \tag{4c}$$
$$F_V = Q_2^2(s) + G_1^2(s), \quad G_{1,3} = -s + \delta_{1,3}K_{1,3}(s),$$

where

$$Q_2(s) = \sqrt{1-s^2}, \quad K_{1,3}(s) = -s \pm \sqrt{r_{1,3}+s^2}. \tag{4d}$$

Here and in the following we use the denotation $r_{1,3} \equiv 1/\delta_{1,3} = \alpha_{1,3}/\alpha_2$. Without loss of generality, the following boundary conditions have been used in the derivation of speed equation (4a): $u(z,t) \to v_1(t)$ if $z \to -\infty$, and $u(z,t) \to v_3(t)$ if $z \to \infty$, where the time-dependent quantities $v_{1,3} = u_{1,3} + \alpha_{1,3}f(t)$ denote zeros of the modified rate function R_F. Consequently, the free front solution $u_0(z)$ satisfies the boundary conditions $u_0(z) \to u_1$ if $z \to -\infty$, and $u_0(z) \to u_3$ if $z \to \infty$. The propagation velocity of the free BF s_0 is described as follows: $s_0 > 0$ if $S > 0$ and $s_0 < 0$ if $S < 0$, where the quantity S denotes the area enclosed by $R-u$ dependence in the interval $[u_1, u_3]$ of the variable u. The presented inequalities show that the penetrated state of the free BF is u_3 if the Maxwellian construction of the rate function is "positively" unbalanced ($S>0$), and differently, the penetrated state is u_1 if $S<0$. From Eq. (4a), in conjunction with Eqs. (4b)–(4d), it follows that $s(t) = s[r_1, r_3, h_R; f^*(t)]$. Whence, the drift velocity of the driven BF, $v \coloneqq \langle s(t) \rangle$, is a function of the slope parameters $r_{1,3}$ and the relative height of the rate function h_R, but not a function of the absolute values of the slope coefficients $\{\alpha_i\}$ and the heights R_M and R_m, if the velocity of the driven BF was taken in the scaled units. Here the bracket $\langle \cdots \rangle$ denotes the averaging over the period of the driving force. In what follows we shall use the scaled units.

As noted, we shall deal with the similarly shaped rate functions defined in above. The Maxwellian construction of the rate function (3) is strictly balanced, i.e., the initial (free) BF is static (motionless) if one takes that $h_R = h_0 \coloneqq -R_M^0/R_m^0$, where the quantities R_M^0 and R_m^0 denote the extreme values of the "balanced," "globally" symmetric rate function, which satisfies the condition $S=0$. The ratio of the extreme values of the considered rate function, R_M^0 and R_m^0, is given by the expression $h_0 = \sqrt{(1+\delta_3)/(1+\delta_1)}$. Using the balance factor g_H defined by the relation $g_H \coloneqq h_R/h_0$ we get that $s_0 = 0$ if $g_H = 1$, i.e., the free BF is always static if the balance factor equals unity. Using the balance factor g_H we get that $s(t) = s[r_1, r_3, g_H; f^*(t)]$. Thus, the governing parameters that influence on the directed drift of BF are as follows, both the balance factor g_H and the slope parameters $r_{1,3}$ of the rate function, and the driving force $f^*(t)$, scaled in the units of the height ΔR. Clearly, the propagation velocity of the initial (free) BF, $s_0(g_H, r_1, r_3)$, is a function of the balance factor g_H and the slope parameters $r_{1,3}$. Consequently, by tuning the factor g_H we arrive at the family of the similarly shaped rate functions, if the slope parameters $r_{1,3}$ were kept fixed. The relevant cases of the similarly shaped functions (3) may be typified as follows:

$$g_H \geqslant 1, \quad \text{or} \quad s_0 \geqslant 0, \tag{5a}$$

$$g_H \leqslant 1, \quad \text{or} \quad s_0 \leqslant 0. \tag{5b}$$

The moment velocity $s(t)$ satisfies some symmetry properties previously discussed in Ref. [8]. In particular, it was shown in [8] that

$$s[r_1,r_3,g_H;f^*(t)] = -s[r_3,r_1,1/g_H;-f^*(t)]. \quad (6)$$

Whence, both families of the "positively" and "negatively" unbalanced rate functions, described the relations (5a) and (5b), may be treated as completely equivalent ones. Turning back to the case of the symmetrical rate functions $\{R_S\}$ we notice that the outer slope coefficients are equal, thus, we set $\alpha_1=\alpha_3$ throughout this paper. Now it follows that the governing parameters of the unforced drift are the balance factor g_H, the slope parameter $r:=\alpha_{1,3}/\alpha_2$, and the driving force $f^*(t)$. The slope parameter r is a function of both the middle slope coefficient α_2 and the outers ones, $\alpha_{1,3}$. Thus the average characteristics of the driven BFs may be governed in two ways: by tuning the middle slope coefficient and the outer.

Speed equation (4a) is transcendental, thus, the numerical simulations have been used to derive the needed characteristics of the driven BFs. In the present paper we shall deal with the biharmonic ac force that is described by a superposition of two harmonic modes. Let us discuss briefly the peculiarities of the biharmonic driver.

IV. BIHARMONIC DRIVER

The temporal symmetry of the driving force plays a very important role in considerations of the unforced drift, especially in the case of the "additive" (nonparametric) driving. The asymmetrically oscillating ac force of zero-mean may serve as a basic "*impetus*" for the unforced transport generated in the ordinary ratchets and ratchetlike systems performing the unidirectional drift of the self-ordered structures in the spatially uniform media. The biharmonic force described by a superposition of the fundamental mode and its second harmonic is frequently used in theoretical studies of the ordinary and "solitonic" ratchets (e.g., see Refs. [1,5,6]). Furthermore, recent studies carried out within the cubic polynomial and the pseudolinear models of the bistable system showed that the average characteristics of BFs that were driven by the symmetrically and asymmetrically oscillating zero-mean forces are rather different (see Refs. [7,8]). For instance, the unforced drift of BF being under the action of a single-harmonic (symmetrically oscillating) ac force disappeared if the rate function was rigorously symmetric, namely, if the relation $R(u_2-\Delta u)=-R(u_2+\Delta u)$ was fulfilled [7]. Differently, the progressive (accelerated) dc drift of BF that was driven by the biharmonic (asymmetrically oscillating) ac force occurred [8].

Zero-mean force, symmetric and asymmetric one, may be presented by a superposition of the harmonic modes. For simplicity's sake we shall deal with the biharmonic force described by the superposition of the fundamental mode and its nth superharmonic,

$$f(t) = f_0\{\sin[(\omega t-\varphi_0)] + b_F \sin[n(\omega t-\varphi_0)+\Delta\varphi]\}. \quad (7)$$

Here the quantity f_0 denotes the amplitude of the fundamental mode, the parameter b_F stands for the relative strength of the superharmonic mode, and the quantities $\Delta\varphi$ and $\varphi_0=\omega t_0$ indicate the relative and initial phase, respectively. To avoid the confusion we notice that the other (modified) definition of the relative phase, namely, $\Delta\Phi:=\Delta\varphi-(n-1)\varphi_0$, is frequently used in considerations of "soliton-ratchets" (e.g., see Refs. [5,6]). The following relation holds: $\sin[n(\omega t-\varphi_0)+\Delta\varphi]\equiv\sin(n\omega t-\varphi_0+\Delta\Phi)$. Whence, both discussed parameters, $\Delta\varphi$ and $\Delta\Phi$, are completely equivalent. In what follows we shall use $\Delta\varphi$ instead of $\Delta\Phi$. Further, in the considered case of the dissipative (extremely overdamped) system the average characteristics of the driven BFs are independent on the initial conditions, i.e., they do not depend on the initial moment t_0 (see Refs. [7,8]). Thus, we assume in what follows that $\varphi_0=0$.

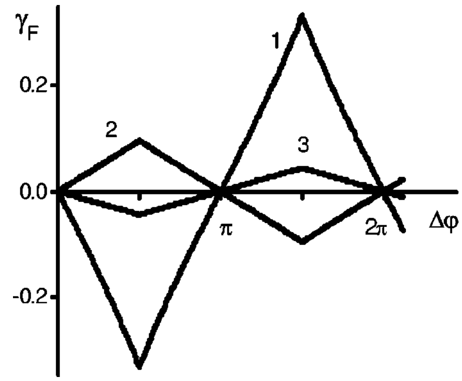

FIG. 2. The dependence of the asymmetry factor of the biharmonic force versus the relative phase $\Delta\varphi$. Parameter values are $b_F=1/2$; (curve 1) $n=2$; (curve 2) $n=4$; (curve 3) $n=6$.

As noted, both cases of the symmetric and asymmetric driver may be presented by the biharmonic function. The degree of the asymmetry of zero-mean force may be evaluated by use of the asymmetry factor γ_F defined by the relation $\gamma_F:=(f_M+f_m)/(f_M-f_m)$, where the quantities $f_M>0$ and $f_m<0$ denote the maximal deviations ("amplitudes") of the driving force. Clearly, a rough estimate of the asymmetry is given by use of the factor γ_F; in general, both the "degree" of the asymmetry of the driving force and the peculiarities of the unforced drift depend on the shape of $f-t$ dependence too. Let us turn to the symmetry properties of the biharmonic driver (7).

The odd harmonic mixing ($n=2m+1$, where $m=1,2,3,\ldots$) describes the rigorously symmetric driver, whereas the other case of the even n's stands for asymmetric one. The obvious relation holds: $\sin[(2m+1)(\varphi+\pi)+\Delta\varphi]=-\sin[(2m+1)\varphi+\Delta\varphi]$. Whence, the contrasymmetricity relation $f(t+T/2)=-f(t)$ is satisfied in any case of the odd mixing. Evidently, in the considered case of the odd n's one has that $\gamma_F=0$, quite similar to the case of the single-harmonic force. Differently, the temporal symmetry of the biharmonic force that involves even the superharmonic mode (even n's) depends on both the relative phase $\Delta\varphi$ and the factor n. More exactly, the asymmetry factor $\gamma_F(\Delta\varphi)$ is the periodic function of $\Delta\varphi$, with the magnitude of the oscillations being decreased with increasing n (see Fig. 2). The following relation holds: $\gamma_F\to 0$ if $n\to\infty$. Furthermore, the asymmetry factor γ_F tends to zero if $\Delta\varphi\to m\pi$. The given relation implies that the biharmonic ac force with even super-harmonic mixing is rigorously contrasymmetric, namely, the relation $f(t-t_N)=-f(t_N-t)$ holds if the relative phase satisfies the equality $\Delta\varphi=\pi,2\pi,3\pi,\ldots$. Here by t_N

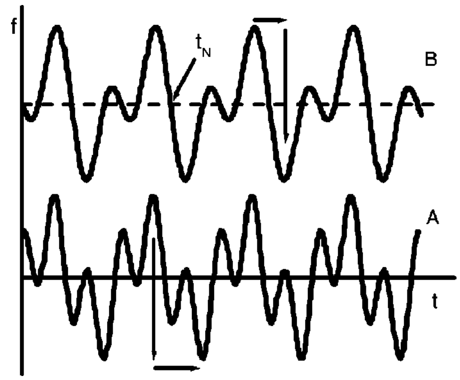

FIG. 3. Temporal symmetry of the biharmonic driver. Two kinds of the contrasymmetric functions $f(t)$ are shown by curves (A) and (B). The parameter values are $\gamma_F=0$, $b_F=1$; (curve A) $n=3$, $\Delta\varphi=\pi/2$; (curve B) $n=2$, $\Delta\varphi=\pi$. Arrows show the symmetry.

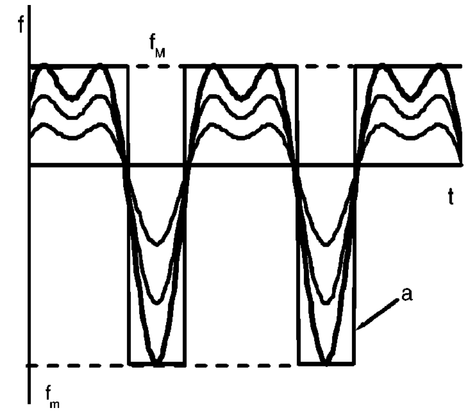

FIG. 4. Similarly shaped functions $f(t)$; the parameter values are $n=2$, $b_F=1/2$, $\Delta\varphi=\pi/2$; $A_F=1/2$, $\gamma_F=-1/3$. The square-pulse approximation (9) is shown by curve (a).

we denote zero-point of the biharmonic function (7). The presented relation $f(t-t_N)=-f(t_N-t)$ stands for another criterion of the "rigorously" symmetric driver. Generally speaking, the contra-symmetry of the ac force implies that one of the following relations: either (a) $f(t+T/2)=-f(t)$ or (b) $f(t-t_N)=-f(t_N-t)$ is satisfied (see Fig. 3). Both given criteria of the temporal symmetry are almost equivalent; the $v-f_a$ characteristics of the "symmetrically" driven BFs are quite similar in both cases discussed (see below, Fig. 5). It should be noted that the single-harmonic function, which describes the ac driver is of the "highest" symmetry, satisfies both the "translational-invariance" condition (a) and the "inverse-symmetry" (with respect to zero-point t_N) property (b).

Turning back to the case of the even n's we notice that the periodicity of the asymmetry factor $\gamma_F(\Delta\varphi)$ implies that both intervals $[0,\pi]$ and $[\pi,2\pi]$ of the variable $\Delta\varphi$ are completely equivalent. Namely, the following relations hold: $A_F(\Delta\varphi)=1/A_F(2\pi-\Delta\varphi)$ and $A_F(\Delta\varphi)=1/A_F(\pi+\Delta\varphi)$, where the quantity $A_F:=-f_M/f_m$ denotes the ratio of the amplitudes. Further, the biharmonic driver consisting of the superposition of the fundamental mode and its second harmonic is of the highest asymmetry. The maximal asymmetry factor γ_F is achieved by taking $n=2$ and $\Delta\varphi=\pi/2$ (or equivalently, $\Delta\varphi=3\pi/2$). For the considered case of the highest asymmetry the dependence of the asymmetry factor versus the relative amplitude b_F is described by the following expressions:

$$\gamma_F=\frac{A_F-1}{A_F+1}, \quad A_F=\frac{b_F^2+1/8}{b_F(1+b_F)}. \tag{8}$$

From Eq. (8) it follows that the extreme, absolutely maximal asymmetry of the biharmonic driver is achieved by taking $b_F=1/2$. Namely, one has that $\gamma_F=-1/3$ and $A_F=1/2$, by taking $\Delta\varphi=\pi/2$. Quite similar, in the other case of the "opposite" asymmetry, defined by the relation $\Delta\varphi=3\pi/2$, one gets that $\gamma_F=1/3$ and $A_F=2$. Finally, the dependence of the asymmetry factor versus the relative amplitude b_F is non-monotone. For instance, one gets that $\gamma_F(b_F=1)=7/25$ and $\gamma_F(b_F=1/4)=1/4$, by taking $n=2$, and $\Delta\varphi=\pi/2$.

We conclude by noting that the asymmetry of the biharmonic force is relatively low; the maximal ratio of the amplitudes f_M and $|f_m|$ does not exceed the value 2. Moreover, the asymmetry factor decreases very rapidly with the increasing n, as shown in Fig. 2. For instance, for the "highest" asymmetry case, given by the relations $\Delta\varphi=\pi/2$ and $b_F=1/2$, one gets that $\gamma_F(n=2)=-1/3$, $\gamma_F(n=4)\approx 0.1$, $\gamma_F(n=6)\approx -0.04$, etc. The additional extra-modes should be included into expression (7) to achieve the higher asymmetries of the driving force. Nevertheless, the analytic treatment of the directed drift generated by the multiharmonic force seems to be complicated. Similarly as in Refs. [7,8], to achieve the needed asymmetries of the driving force we have used the square-pulse "approximation" described by expression (see Fig. 4)

$$f(t)=\begin{cases} f_M, & nT<t<nT+T_M,\\ f_m, & nT+T_M<t<(n+1)T. \end{cases} \tag{9}$$

As previously, the quantities f_M and f_m denote the amplitudes and the variable parameters T_M and T_m indicate the "half-periods" of the driving force, namely, one has that $T=T_M+T_m$. To satisfy zero-mean condition we demand that $T_m/T_M=A_F$, where by A_F we denote the ratio of the amplitudes, $A_F=-f_M/f_m$. The asymmetry of the square-wave function (9) is arbitrary: by tuning the free parameters T_M and T_m within the interval $(0,\infty)$ one gets $0<A_F<\infty$, or differently, $-1<\gamma_F<1$.

The strength of the asymmetric driver depends on the amplitudes (maximal deviations) f_M and f_m. When considering $v-f_a$ characteristics we shall use the average amplitude $f_a:=(f_M-f_m)/2$, which indicates the "effective" strength of the driving force. Further, in what follows we shall deal with the *similarly shaped* functions $f(t)$; the ratio of the amplitudes f_M and f_m will be kept fixed, when the strength of the driving force is changed. The $v-f_a$ characteristics presented below have been derived using the similarly shaped functions $f(t)$. Turning back to the biharmonic driver we notice that the strength of the biharmonic ac force depends on the parameters f_0, b_F, $\Delta\varphi$, and the integer n. By tuning the amplitude of the fundamental mode f_0 one arrives to the family of the similarly shaped functions $\{f(t;f_0)\}$, if the rest parameters are kept fixed (see Fig. 4). Once again, both the ratio of the amplitudes A_F and the asymmetry factor γ_F are fixed within the considered family of the similarly shaped functions. Let us turn to the driven BFs, let us discuss the characteristic features of the unforced drift generated by the asymmetric driver.

V. UNIDIRECTIONAL DRIFT INDUCED BY ASYMMETRICALLY OSCILLATING ZERO-MEAN FORCE

The unforced dc drift of BF takes place if either the rate function or the driving force breaks the symmetries discussed above. The considered dc drift disappears if both the rate function and the driving force are rigorously symmetric, namely, if the "contrasymmetry" condition $R(u_2-\Delta u) = -R(u_2+\Delta u)$ and the temporal symmetry relation, either $f(t+T/2)=-f(t)$ or $f(t-t_N)=-f(t_N-t)$, are satisfied. In the present paper we shall deal with the case of the broken symmetry above labeled by R_S-f_A. The particular case of the asymmetrically oscillating zero-mean force and the *symmetrical* rate functions $\{R_S\}$ is considered, thus, we take that $r_1=r_3\equiv r$.

The strength of the driving force, the maximal amplitude, either f_M or f_m, cannot exceed the extreme value of the rate function, either R_M or R_m, respectively. The middle zero-point v_2 of the modified rate function R_F and the outer, either v_1 or v_3, closely approach to each other if the amplitude of the similarly shaped function $f(t)$ tends to the critical value, either R_M or R_m. The criteria of the global stability of the driven BF read, $f_M<R_M$ and $|f_m|<|R_m|$. Otherwise, using the scaled unities we demand that $f_M^*<(1+h_R)^{-1}$ and $|f_m^*| < h_R(1+h_R)^{-1}$.

Before discussing the peculiarities of the directed drift generated by the asymmetrically oscillating zero-mean force let us touch briefly on the dc drift induced by the rigorously symmetric driver. As noted, two kinds of the "rigorously" symmetric functions $f(t)$ exist. Furthermore, the biharmonic driver with the odd harmonic mixing is strictly symmetric, in any case. At last, the biharmonic function consisting of a superposition of the fundamental mode and its even superharmonic is "rigorously" symmetric too, if the relative phase was taken as $\Delta\varphi=m\pi$. We already noted that the characteristic features of the unforced drift generated by the rigorously symmetric driver are quite similar, no matter what kind of the "rigorous" symmetry, either $f(t+T/2)=-f(t)$ or/and $f(t_N-t)=-f(t_N+t)$, was satisfied. The typical $v-f_a$ characteristics of the symmetrically driven BFs, followed from speed equation (4a), are shown in Fig. 5. All above discussed versions of the symmetric driver are presented. These are (a) a single-harmonic force (curve 1), (b) the biharmonic force with the odd harmonic mixing (curves 2, 2a and 3), and (c) the biharmonic force with the even superharmonic mixing (the particular case of the superharmonic mode, $n=2$ and $\Delta\varphi=\pi$) (curve b). It is easy to see that the presented $v-f_a$ characteristics are very similar to those given in the case of the single-harmonic force (see also Refs. [7,8]). The progressive (accelerated) dc drift of the initially propagating ($s_0 \neq 0$) BFs was occurred, namely, the magnitude of drift velocity $|v|$ monotonically increases with the increasing amplitude f_a. More specifically, the unforced drift, the shift of the mean velocity of BF takes place if the Maxwellian construction of the rate function $R(u)$ is unbalanced, i.e., if the balance factor g_H satisfies the inequality $g_H\neq 1$. Differently, the motionless BF, which stays initially at rest, cannot gain the dc motion discussed, in each case of the symmetric driver, regardless to the peculiarities of $f-t$ dependence (see the straight thick line in figure). Thus we conclude that the directed drift disappears if both the rate function and the ac driver are rigorously symmetric, namely, if the equalities $R(u_2-\Delta u)=R(u_2+\Delta u)$ and, either $f(t+T/2)=-f(t)$ or/and $f(t_N-t)=-f(t_N+t)$, are fulfilled. This conclusion was supported by our direct calculations that have been performed using various "combinations" of the symmetrically oscillating force (7). It is interesting to note that a similar result was recently obtained in the case of the topological soliton that was driven by biharmonic ac force (7); the directed motion of the sine-Gordon kink being under the action of the biharmonic force consisting of the superposition of the fundamental mode and its third superharmonic ($n=3$) disappeared (see Ref. [5]). Clearly, the "rate function" (nonlinearity) of sine-Gordon equation is rigorously symmetric.

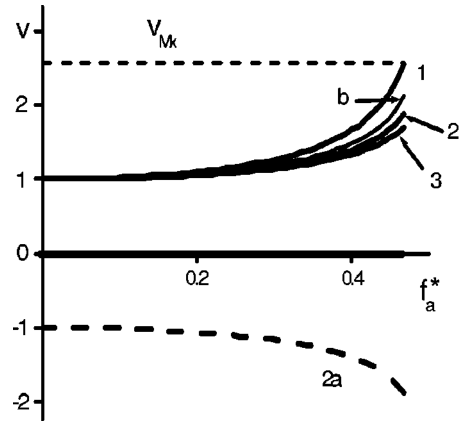

FIG. 5. The dependence of the drift velocity (arbitrary units) of the symmetrically driven BF versus the amplitude f_a. The driving forces are (curve 1) the single-harmonic force, $n=1$; (curves 2, 2a, and 3) the case of the odd harmonic mixing: $n=3$ and $n=5$, respectively; (curve b) the case of the even superharmonic: $n=2$ and $\Delta\varphi=\pi$. The parameter values are $r=0.2$, $\gamma_F=0$; (curves 1, 2, 3, b) $g_H=1.1$; (curve 2a) $g_H(2a)\equiv 1/g_H(2)\approx 0.91$. Other parameter values are (curves 2, 2a, 3) $b_F=0.5$, $\Delta\varphi=\pi/2$; (curve b) $b_F=0.5$. The contra-symmetric curves (2) and (2a) illustrate the symmetry property (6).

In continuing the discussion of the symmetrically driven BFs we emphasize that the considered $v-f_a$ dependencies are very close to each other in any case of the "odd mixing" ($n=3,5,7,9,\ldots$), if the rest parameters of both the driving force (f_0, b_F, and $\Delta\varphi$) and the rate function (r and g_H) were taken fixed. Broadly speaking, the relative deviation $\Delta v/v_{Mx}$ does not exceed the few percent if one takes that $n>5$, where by Δv we denote the maximal deviation of the drift velocity and the quantity $v_{Mx}:=v(f_{Mx})$ indicates the drift velocity taken at the maximal driving force. The steepness of $v-f_a$ characteristics depends on the slope parameter r. The considered characteristics become more and more flattened if

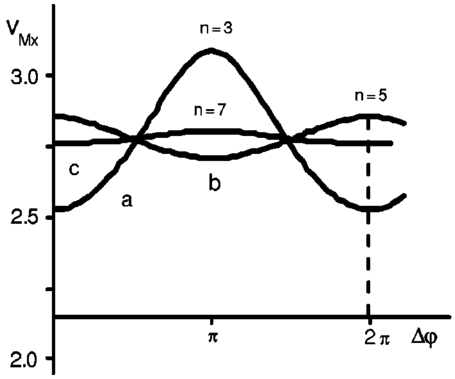

FIG. 6. The dependence of the maximal drift velocity (arbitrary units) of the symmetrically driven BF versus the relative phase $\Delta\varphi$. The parameter values are $r=0.2$, $g_H=5/4$; $\gamma_F=0$, $b_F=0.5$; (curve *a*) $n=3$, (curve *b*) $n=5$, (curve *c*) $n=7$.

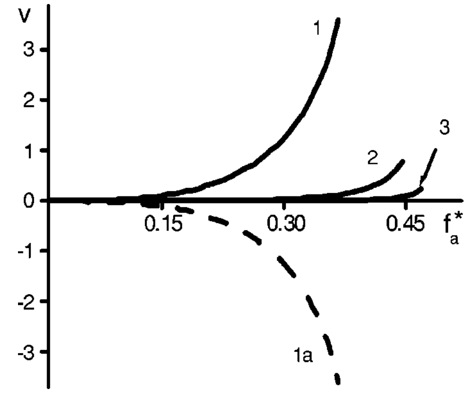

FIG. 7. The dependence of the drift velocity (arbitrary units) of the asymmetrically driven BF versus the amplitude f_a. The parameter values are (curves 1 and 1*a*) $n=2$; (curves 2, 3) $n=2,6$, respectively. The other parameter values are $r=0.2$, $g_H=1$, $b_F=0.5$; (curves 1 and 3) $\Delta\varphi=\pi/2$; (curves 2 and 1*a*) $\Delta\varphi=3\pi/2$. The asymmetry factors are (curve 1) $\gamma_F=-1/3$; (curve 2) $\gamma_F\approx-0.1$; (curve 3) $\gamma_F\approx-0.04$; (curve 1*a*) $\gamma_F=1/3$.

the slope parameter r increases, quite similar as in the other case of the asymmetrically shaped rate functions $\{R_A\}$ (see Ref. [8]). The "size" of the driving effect, the maximal drift velocity v_{Mx} of the symmetrically driven BF depends on the relative phase $\Delta\varphi$ that governs the shape of $f-t$ dependence. The dependence of the maximal drift velocity versus the relative phase is shown on Fig. 6. The presented $v_{Mx}-\Delta\varphi$ dependencies show that the maximal drift velocity v_{Mx} is the periodic function of $\Delta\varphi$, with the amplitude of the oscillations decreasing with the increasing n. The considered dependencies become very flattened if the integer n is large enough, namely, if the relation $n=2(m+1)\gg 1$ was satisfied (see curve c in figure). Hence, the "efficiency" of the rigorously symmetric driver satisfying the relation $n\gg 1$ is nearly constant, i.e., it does not depend on the frequency of the "supermode" (within the quasistatic approximation used). Referring to the other case of the nearly symmetric driver, defined by the relations $n=2m\gg 1$ and $\gamma_F\ll 1$, we note that $v-f_a$ characteristics of the "almost" symmetrically driven BFs are very similar to those given in the case of the rigorously symmetric driver. Small deviations between two discussed types of $v-f_a$ characteristics occur only in the limiting region of the high strengths of the driving force, when the amplitude f_a approaches to the critical value f_{Mx} (see below, Fig. 7). The considered deviation vanishes in the limit $n\to\infty$. Roughly speaking, all biharmonic drivers satisfying the relation $n\gg 1$ may be treated as symmetric ones, no matter what superharmonic mode, odd or even, was involved.

In closing the discussion of the symmetrically driven BFs we summarize that the progressive (accelerated) dc drift is generated by the symmetrically oscillating force if the rate function is *symmetrical* and "unbalanced," i.e., if Maxwellian construction of the rate function was not balanced. The unforced dc drift disappears if both the rate function and the driving force are rigorously symmetric, namely, if the following relations hold: (a) $R(u_2-\Delta u)=R(u_2+\Delta u)$ and (b) $f(t+T/2)=-f(t)$ or/and $f(t-t_N)=-f(t_N-t)$. The considered dc drift is more pronounced, namely, $v-f_a$ characteristics of are more rapid if the slope parameter r is lesser; the steepness of the characteristics increases if the inner (outer) slope coefficient of the rate function increases (decreases).

The main conclusions concerning the discussed case of the directed drift have been confirmed by our direct calculations that have been performed by use the cubic polynomial (*symmetrical*) rate functions $R_K=a(u-w_1)(u-w_2)(u-w_3)+C$ satisfying the relation $w_3-w_2=w_2-w_1>0$. It is interesting to note that $v-f_a$ dependencies given in both cases of the cubic polynomial and pseudolinear rate functions are very close to each other if the slope coefficients of pseudolinear function (3) are taken in accordance with the relation $\alpha_i=R'_K(w_i)$ (clearly, both the drift velocity v and the amplitude f_a should be taken is the scaled units defined above). The peculiarities of the directed drift generated by the rigorously symmetric driver and the *symmetrical* rate functions $R_S(u)$ will be discussed more inclusively elsewhere. Let us turn to the other case of the even n's; let us discuss the unforced drift of the asymmetrically driven BFs.

As noted, the asymmetry of the biharmonic driver consisting of a superposition of the fundamental mode and its even superharmonics depends on both the relative phase $\Delta\varphi$ and the integer n, which indicates the frequency of the even mode. The asymmetry factor γ_F decreases very rapidly with the increasing n; the asymmetry factor γ_F does not exceed the value of few tenths if one takes that $n=2m>4$ (see Fig. 2). As discussed, the average characteristics of BFs being under the action of the biharmonic force consisting of a superposition of the fundamental mode and its higher superharmonics are quite similar to those given in the case of the rigorously symmetric driver. The $v-f_a$ characteristics shown in Fig. 7 illustrate the dependence of the "size" of the driving effect versus the frequency (integer n) of the even superharmonic mode. The particular case of the rigorously symmetric rate function $R(u_2-\Delta u)=R(u_2+\Delta u)$ is presented. The given characteristics demonstrate that the progressive dc drift of the initially static BFs being under the action of the asymmetrically oscillating zero-mean force takes place. As expected, the "size" of the driving effect decreases very rapidly with the increasing n; the $v-f_a$ dependencies become more and more flattened if the frequency of the superharmonic mode increases (compare curves 1, 2, and 3). The presented dependencies evidently show that the maximal driving effect is achieved with the highest asymmetry force defined by the relations $n=2$, $b_F=1/2$ and $\Delta\varphi=(m+1/2)\pi$ (see curves 1 and 1a). The considered characteristics that are related to the higher super-modes $n>4$ are very flattened; the dc drift is practically disappeared in the whole interval of the amplitudes f_a, except the narrow interval lying in the vicinity of the extreme amplitudes, $f_a \cong f_{Mx}$ (see curve 3 in Fig. 7). At last, the propagation direction of the asymmetrically driven BF that is described by the rigorously symmetric rate function depends on the sign of the asymmetry factor γ_F. The following relations hold: $v>0$ if $\gamma_F<0$ and $v<0$ if $\gamma_F>0$. The presented inequalities imply that the modified Maxwellian rule holds: $v>0$ if $S_a>0$ and $v<0$ if $S_a<0$, where by S_a we denote the total area enclosed by the rate function $R_a \equiv R(u)-f_a$. Clearly, the given inequalities are in agreement with the symmetry relation $s(r,g_H;f^*)=-s(r,1/g_H;-f^*)$ followed directly from Eq. (6) (compare curves 1 and 1a).

It should be noted that the progressive dc drift of the initially static BF takes place too in the other case of the "opposite" R_A-f_S symmetry, discussed in Ref. [8]. Moreover, the progressive dc drift generated in the case of the "opposite" symmetry satisfies the relations $v>0$ if $h_R>1$ and $v<0$ if $h_R<1$. To draw some closer parallels between two discussed cases of the asymmetry, labeled by R_S-f_A and R_A-f_S, we introduce the factor $k_{FR}:=A_F/h_R$, which indicates the ratio of "asymmetries" of both the driving force and the rate function. The obvious relations hold: $A_F>h_R$ if $k_{FR}>1$ and $A_F<h_R$ if $k_{FR}<1$. Now, the "modified rule" that encompasses both cases discussed reads $v<0$ if $k_{FR}>1$ and $v>0$ if $k_{FR}<1$. The unforced drift generated by the balanced rate function, either *symmetrical* (R_S) or *asymmetrical* (R_A), is always directed toward the stabile state u_1 if the inequality $k_{FR}>1$ holds, and differently, the penetrated state of the initially static BF is u_3 if $k_{FR}<1$. Clearly, the limiting case given by the relation $k_{FR}=1$ stands for the "totally" symmetric DFR, labeled by R_S-f_S. Both the rate function and the driving force are rigorously symmetric in such a case; thus one gets $v=0$, in accordance with the "modified rule." Thus we conclude the asymmetric driving plays a quite similar role as the asymmetry of the rate function; the characteristic features of the directed drift generated in both cases of the "opposite" asymmetry are quite similar.

Turning back to the unforced drift of the asymmetrically driven BF, generated by the rigorously symmetric rate function, $R(u_2-\Delta u)=R(u_2+\Delta u)$, we note that the "size" of the driving effect, the maximal drift velocity v_{Mx}, which indicates the steepness of $v-f_a$ characteristics, depends on both the slope parameter r and the relative phase $\Delta\varphi$. The dependence of the maximal drift velocity versus the relative phase is shown in Fig. 8, for the different values of the integer n. As expected, the maximal drift velocity is periodic function of $\Delta\varphi$, in accordance with $\gamma_F-\Delta\varphi$ dependence discussed above (see Fig. 2). The "stopping" points S shown by arrows on Fig. 8 satisfy the relation $v_{Mx}(\Delta\varphi_S)=0$. The following relation holds: $\Delta\varphi_S \equiv m\pi$. As discussed, the biharmonic ac force with the "even mixing" is rigorously symmetric ($\gamma_F=0$) if one takes that $\Delta\varphi=\Delta\varphi_S$. Thus, the directed drift will

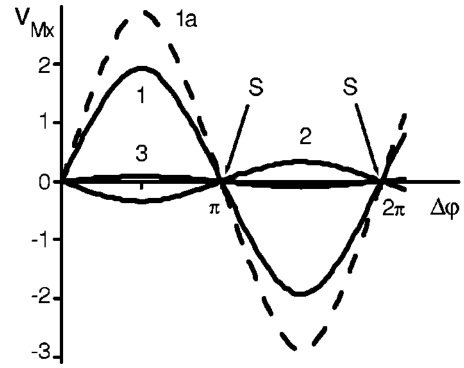

FIG. 8. The dependence of the maximal drift velocity (arbitrary units) of the initially static BF versus the relative phase $\Delta\varphi$. The parameter values are $g_H=1$, $b_F=0.5$; (curves 1, 2, 3) $n=2,4,6$, respectively, and $r=1$; (curve 1*a*) $r=0.1$ and $n=2$.

disappear at any strength of the driving force if one takes that $\Delta\varphi=\Delta\varphi_S$. Further, the maximal drift velocity decreases very rapidly with the increasing n, as shown by curves 1, 2, and 3; the dc drift practically disappears if one takes that $n>4$. Furthermore, the presented curves 1 and 1a confirm our previous conclusion: the maximal driving effect is achieved with the highest asymmetry force satisfying the relations $n=2$ and $\Delta\varphi=(m+1/2)\pi$. At last, $\gamma_F-\Delta\varphi$ dependencies shown by curves 1 and 1a evidently show that the driving effect is more strongly pronounced at the lesser value of the slope parameter r. Whence, $v-f_a$ characteristics taken at the larger (smaller) value of the inner (outer) of the slope coefficient α_i will be steeper.

We already noted that the asymmetry of the biharmonic force is relatively low; more exactly, the following relations are satisfied: $-1/3<\gamma_F<1/3$ and $0.5<A_F<2$. Differently, the asymmetry of the square-pulse function (9) is flexible; the following relations hold: $-1<\gamma_F<1$ and $0<A_F<\infty$. Moreover, the size of the driving effect depends on the shape of $f-t$ dependence. The maximal driving effect is achieved with the periodic functions $f(t)$ that are characterized by the flattened profiles in the vicinity of the maximal deviations (amplitudes) f_M and f_m (see Refs. [7,8]). Thus the periodically oscillating square-pulse driver of zero-mean seems to be more "effective," if compared to a biharmonic one. The "efficiency" of both drivers, described by the biharmonic and the square-wave functions, is demonstrated by $v_{Mx}-\gamma_F$ dependencies shown in Fig. 9. The presented $v_{Mx}-\gamma_F$ dependence shown by curve *a* corresponds to the biharmonic driver of the highest asymmetry ($\gamma_F=1/3$). As expected, the efficiency of the square-pulse force is much more higher. The ratio of the maximal drift velocities v_{Mx} taken at the extremes of both curves (b) and (a) approximately equals 4. In addition, both curves (b) and (a) are contrasymmetrically shaped ones, namely, the relation $v_{Mx}(-\gamma_F)=-v_{Mx}(\gamma_F)$ holds, in agreement with the symmetry relation (6). Nevertheless, the character of the considered dependences is rather different; $v_{Mx}-\gamma_f$ dependence shown by curve *b* is nonmonotone, in contrast to that given by curve *a*. The extreme (absolutely maximal) value of the velocity v_{Mx} is achieved with the square-pulse function of the "optimal" asymmetry described by the relation $\gamma_F=\gamma_{\mp}\approx\mp 0.2$. We emphasize, that the "op-

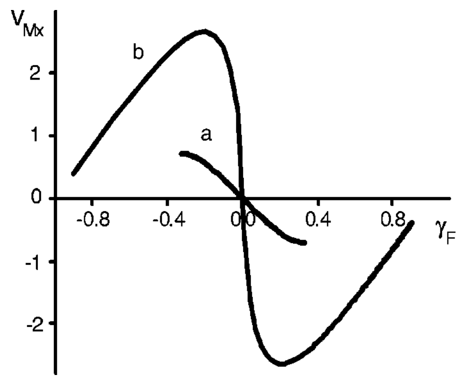

FIG. 9. The dependence of the maximal drift velocity (arbitrary units) of the initially static BF versus the asymmetry factor γ_F. The driving forces are square-pulse force (9) (curve b); biharmonic force (7) (curve a). The parameter values are $g_H=1$, $r=0.2$, and $b_F=0.5$. Two cases of the "opposite" asymmetry of the biharmonic force, $\gamma_F<0$ and $\gamma_F>0$, correspond to the different intervals of the relative phase $\Delta\varphi:[0,\pi]$ and $[\pi,2\pi]$, respectively.

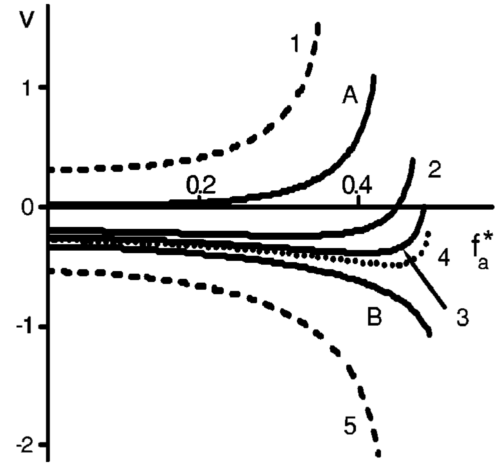

FIG. 10. The dependence of the drift velocity (arbitrary units) of the asymmetrically driven BF versus the amplitude f_a. The parameter values are $r=0.1$, $A_F=0.7$, and $\gamma_F\approx-0.18$; curves (1, A, 2, 3, 4, B, 5) $g_H\approx 1.40, 1.0, 0.80, 0.75, 0.73, 0.70, 0.55$, respectively; (curve B) $k_{FR}=1$.

timal" values of the asymmetry factor, γ_- and γ_+, do not depend of the slope parameter r. Differently, the maximal drift velocity, namely, the magnitude $|v_{Mx}(\gamma_\mp)|$, taken at the "optimal" asymmetry of the ac force increases with the decreasing r. As noted, the $v_{Mx}-\gamma_F$ dependence related to the biharmonic force (curve a) is monotone; the maximal driving effect is achieved with the highest asymmetry force satisfying the relation $|\gamma_F|=1/3$.

In closing the discussion of the unforced drift generated by the rigorously symmetric rate functions, $R(u_2-\Delta u)=R(u_2+\Delta u)$, we conclude that the progressive dc drift of the asymmetrically driven BFs occurred. The size of the driving effect, and the shift of the mean velocity of the ac driven BF depends on both the slope parameter r of the rate function and the asymmetry factor γ_F that characterizes the asymmetry of the driving force. The drift velocity (magnitude) of the unforced drift of BF increases with the decreasing slope parameter r. The maximal driving effect of BF being under the action the biharmonic ac force consisting of a superposition of the fundamental mode and its even superharmonic is achieved with the highest asymmetry ac force satisfying the relations $n=2$, $b_F=1/2$ and $\Delta\varphi=(m+1/2)\pi$. The increase of the frequency of the even super-harmonic mode shrinks the driving effect, i.e., the drift velocity of BF decreases with the increasing n. Finally, the size of the driving effect depends on the peculiarities of $f-t$ dependence. The unforced drift of BF, generated by the square-pulse ac force is much more strongly pronounced, if compared to that given in the case of the biharmonic ac force. The drift velocity of BF being under the action of the square-pulse driving is a nonmonotonic function of the asymmetry factor γ_F; the "optimal" asymmetry of the considered zero-mean force, which induces the maximal driving effect, exists. The maximal driving effect is achieved if the asymmetry factor, the square-pulse driver, was taken as follows: $\gamma_F\approx\mp 0.2$.

Let us turn to the other case of the initially propagating BFs being under the action of the asymmetrically oscillating zero-mean force. Differently as in the previous case of the "globally" symmetric rate function, the Maxwellian construction of the rate function is not balanced in this case, namely, the inequalities $h_R\neq h_0$ and $g_H\neq 1$ hold. As earlier, we shall deal with the symmetrically shaped rate functions $\{R_S\}$.

From Eq. (6) it follows that the dc drift generated by the similarly shaped rate functions satisfy the symmetries

$$v(r,g_H;\gamma_F,f_a)=-v(r,1/g_H;-\gamma_F,f_a), \tag{6a}$$

$$v(r,g_H;k_{FR},f_a)=-v(r,1/g_H;1/k_{FR},f_a), \tag{6b}$$

if one keeps in mind that the two contrasymmetric forces of the opposite asymmetry satisfy the relation $f(t;\gamma_F,f_a)=-f(t;-\gamma_F,f_a)$. The presented equalities (6a) and (6b) show that the dc drift generated in both cases of the "positively" ($g_H>1$) and "negatively" ($g_H<1$) unbalanced rate functions is corelated. More specifically, $v-f_a$ characteristics generated by two "oppositely" unbalanced rate functions, $R_I(u;r,g_H)$ and $R_{II}(u;r,1/g_H)$, will be contrasymmetrically shaped ones, if the driving forces were taken contrasymmetric ones, namely, if the relation $f_I(t;\gamma_F,f_a)=-f_{II}(t;-\gamma_F,f_a)$ was satisfied (see Fig. 11 below).

Keeping in mind symmetry relations (6a) and (6b), let us begin with directed drift induced by the ac force of the "negative" asymmetry, defined by the relation $\gamma_F<0$. The case of the opposite asymmetry, $\gamma_F>0$, will be discussed below (see Fig. 11). As noted, the asymmetry of the biharmonic force is low. Differently, the asymmetry of the ac square-wave function (9) is flexible. The periodic square-pulse force of zero-mean will provide us with a generic picture of the unforced drift generated by the driving force of the arbitrary asymmetry. The typical $v-f_a$ characteristics of the initially propagating BFs that are driven by the asymmetric square-pulse force of the negative asymmetry ($\gamma_F<0$) are presented in Fig. 10, for a wide interval of the balance factors g_H. As previously, $v-f_a$ dependencies shown in Fig. 10 are quite similar to those given in the case of the opposite

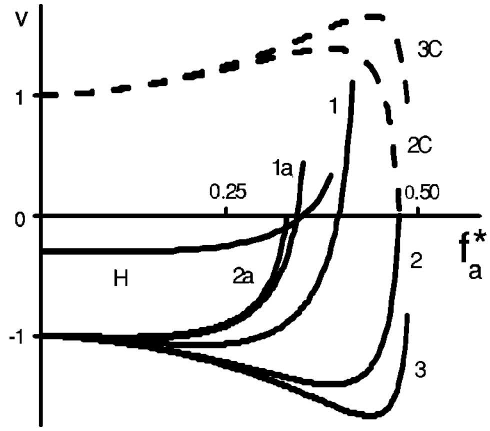

FIG. 11. The dependence of the drift velocity (arbitrary units) versus the amplitude of the driving force. The parameter values are $r=0.1$; (solid curves) $g_H=0.7$; (dashed curves $2C$ and $3C$) $g_H \approx 1.4$. The other parameter values are (curves 1, 2, 3) $\gamma_F \approx -0.41, -0.23, -0.20$, respectively; (curves $1a$, $2a$) $\gamma_F \approx -0.67, -0.78$, respectively, (curves $2C$ and $3C$) $\gamma_F(2C)=-\gamma_F(2)\approx 0.23$ and $\gamma_F(3C)\approx 0.20$; (curve H) $g_H=0.9$, $n=2$, $b_F=1/2$, $\Delta\varphi=\pi/2$, $\gamma_F=1/3$.

symmetry R_A-f_S (see Ref. [8]). The characteristic features of the unforced drift generated in both cases of the DFR satisfying the symmetries R_S-f_A and R_A-f_S practically coincide (see Fig. 3 in Ref. [8]). It is easy to see that $v-f_a$ dependencies shown in Fig. 10 confirm our previous conclusion: both the asymmetric driver and the *asymmetrical* (asymmetrically shaped) rate function play quite similar role. More specifically, two families of the differently shaped $v-f_a$ characteristics, monotone and nonmonotone ones, shown in the figure are separated by the "marginal" curves A and B satisfying the relations $g_H=1$ (or differently, $k_{FR}=A_F$) and $g_H=A_F$ (otherwise, $k_{FR}=1$). Both of these are related to the different types of the unforced migration. More exactly, monotone curves 1 and 5 satisfying the relations (a) $dv/df_a>0$ and (b) $dv/df_a<0$, respectively, show that the progressive dc drift of BF takes place. Namely, the magnitude of the drift velocity (the absolute value, $|v|$) of BF is monotonically increasing function of the amplitude f_a. Both discussed kinds of $v-f_a$ dependencies that describe the progressive drift satisfy the relations (a) $g_H>1$ (shown by curve 1) and (b) $g_H<A_F$ (shown by curve 5). Referring to the other case of the negative asymmetry, $\gamma_F>0$, we notice that the progressive type of the directed motion satisfies the reciprocal inequalities, namely, $g_H<1$ and $g_H>A_F$, in accordance with symmetry relations (6a) and (6b). As noted, $v-f_a$ dependencies generated in both discussed cases of the opposite asymmetry, $\gamma_F<0$ and $\gamma_F>0$, are contra-symmetrically shaped ones (see below Fig. 11).

The intermediate characteristics (shown by curves 2, 3, and 4 in Fig. 10) that are located in-between the limiting curves A and B satisfy the relation $A_F<g_H<1$. Similarly as in the case of the "opposite" symmetry R_A-f_S, different types of "nonprogressive" drift were occurred. These are as follows: the reversal type of the directed drift (curve 2), the stopping of the initially propagating BF (curve 3), and the "regressively-progressive" motion shown by nonmonotonic curve 4. Quite similar to the case of R_A-f_S symmetry, both the stopping of the initially propagating front, that is driven at the maximal driving force, and the reversal of the directed motion of BF take place. The wideness of the intermediate region, the interval of the balance factors Δg_H corresponding to the intermediate characteristics of "nonprogressive" type depends on the asymmetry of the driving force. The following relation holds: $\Delta g_H=A_F-1$. More exactly, the balance factors g_H of the rate functions that generate "nonprogressive" drift are described as follows: $g_H\in(1,\Delta g_H)$. The following relations hold: $\Delta g_H>0$ if $\gamma_F>0$ and $\Delta g_H<0$ if $\gamma_F<0$. Thus it follows that $\Delta g_H\to 0$ if $\gamma_F\to 0$; the intermediate region vanishes in the limiting case of the rigorously symmetric driver. More specifically, the intermediate $v-f_a$ characteristics become more and more flattened if the asymmetry factor tends to zero. This conclusion is in good agreement with our previous result discussed above: nonprogressive dc drift was absent in any case of the "rigorously" symmetric ac force (see Fig. 5). Furthermore, the increase of the asymmetry of the driving force, or differently, the decrease of the asymmetry factor γ_F (negative) shifts the "marginal" curve B toward the lesser values of the balance factor g_H. As a consequence, the shape of both the "reversal" and "stopping" characteristics, shown by curves 2 and 3 in Fig. 10, is slightly modified if the asymmetry of the driving force was increased. More exactly, the negative slope of both the "reversal" and "stopping" $v-f_a$ characteristics decreases with the increasing asymmetry of the of the ac force, until the derivative dv/df_a becomes positively defined in the whole interval of the "argument" f_a. Two different kinds of the "reversal" and "stopping" characteristics are shown by curves (1, 2) and $(1a,2a)$ in Fig. 11. One can see that the character of $v-f_a$ dependencies taken in both cases of the low ($|\gamma_F|<0.5$; curves 1 and 2) and high ($|\gamma_F|>0.5$; curves $1a$ and $2a$) asymmetry of the driving force is rather different. Regretfully, it was not able to receive the rigorous criteria, to derive the "critical" values of the parameters responsible for the "transition" from the monotone $v-f_a$ characteristics to nonmonotonic ones, even in the case of the square-pulse force. The considered "transition" depends on many factors: the degree of the disbalance of Maxwellian construction, the shape and the asymmetry factor of the driving force, etc. In closing the discussion of the intermediate characteristics we note that the reversal type of the unforced dc drift of BFs shows that the bistable system being under the action of the asymmetrically oscillating zero-mean force could arrive at a nontrivial behavior, even in the case of the *symmetrical* (symmetrically shaped) rate function: the "phase transition" from the less stable state toward more stable one was occurred. It is interesting to note that the reversal type of the directed motion of BFs takes place too in the previously discussed case of the opposite asymmetry labeled by R_A-f_S (see Ref. [8]).

We already noted that $v-f_a$ characteristics satisfying symmetry relations (6a) and (6b) are contrasymmetrically shaped ones. This property is demonstrated by $v-f_a$ dependencies shown by curves (2,3) and $2C,3C$) in Fig. 11. The presented dependencies that have been derived by use of the square-pulse function (9) satisfy the relations $g_H(2C,3C)=1/g_H(2,3)$ and $\gamma_F(2C,3C)=-\gamma_F(2C,3C)$, and are rigor-

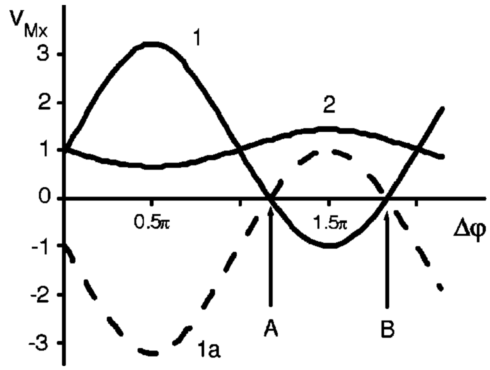

FIG. 12. The dependence of the maximal drift velocity (arbitrary units) of BF versus the relative phase $\Delta\varphi$. The parameter values are $r=0.2$, $b_F=0.5$; (curves 1, 1a) $n=2$, $g_H(1a)=1/g_H(1)=0.95$, $\gamma_F(1a)=-\gamma_F(1)\approx 0.33$; (curve 2) $n=4$, $\gamma_F\approx 0.10$. The stopping points A and B (shown by arrows).

ously contrasymmetric ones, in accordance with Eqs. (6a) and (6b).

As discussed, the progressive dc motion and the other types of the spurious drift described by "intermediate" $v-f_a$ characteristics take place in both cases of the high and low asymmetry of the driving force that was approximated by the square-pulse function. Nevertheless, the intermediate characteristics become very flattened if the asymmetry of the driving force is low. In addition, the steepness of $v-f_a$ characteristics depends on the shape of $f-t$ dependence; the square-pulse ac force is more "effective" if compared to biharmonic one. Finally, the maximal asymmetry factors of both the biharmonic and the square-pulse functions are rather different. As a consequence, $v-f_a$ dependencies generated by the biharmonic ac force are much more flattened if compared to those given in the case of the square-pulse function. This conclusion is illustrated by curve H in Fig. 11. The presented $v-f_a$ dependence shown by curve H was derived with the highest asymmetry force described by the relations $n=2$, $b_F=1/2$, $\Delta\varphi=\pi/2$, and $\gamma_F=-1/3$. As expected, the considered dependence is more flattened if compared to those given by curves 1 and 1a. The lesser values of the disbalance factor $\Delta g_0:=|g_H-1|$ are required to achieve the reversal drift of BF that is driven by the biharmonic force. For instance, the ratio of the factors Δg_0 related to both curves 1 and H approximately equals 4, i.e., one has that $\Delta g_0(1)/\Delta g_0(H)\approx 4$. This implies that the reversal of the BF being under the action of the biharmonic ac force may be achieved if the deviation of Maxwellian construction from the strictly balanced situation is relatively small, or differently, if the initial velocity of BF was small enough. The steepness of $v-f_a$ characteristics related to the biharmonic driver depends on both the slope parameter r and the relative phase $\Delta\varphi$ that governs both the shape and the asymmetry of the driving force. As previously, the considered characteristics are more rapid if the slope parameter r is lesser. The dependence of the maximal drift velocity, which indicates the steepness of $v-f_a$ characteristics, versus the relative phase $\Delta\varphi$ is shown in Fig. 12. The $v_{Mx}-\Delta\varphi$ dependences shown by curves 1 and 1a have been derived with the highest asymmetry force satisfying the relations $n=2$, and $b_F=1/2$, whereas the driving force related to the more flattened curve 2 is described as follows, $n=4$, $b_F=1/2$ and $\gamma_F\equiv 0.1$. The presented curves show that the maximal drift velocity v_{Mx} is the periodic function of $\Delta\varphi$, in agreement with $\gamma_F-\Delta\varphi$ dependencies discussed in above. Moreover, the presented curves 1 and 1a that satisfy the relations $g_H(1a)=1/g_H(1)$ and $\gamma_F(1a)=-\gamma_F(1a)$ are contrasymmetrically shaped ones, in accordance with the symmetry relations (6a) and (6b). The driving forces related to the curves 1 and 1a are contrasymmetrically shaped, namely, the equality $f_1(t)=-f_{1a}(t-T/2)$ holds. Furthermore, zero points A and B (shown by arrows in figure) of both characteristics 1 and 1a strictly coincide. This implies the "critical" strengths of the driving forces that induce the stopping (reversal) of the ac driven BF strictly coincide if the rate functions and the driving forces are contrasymmetrically shaped ones, namely, if one takes that (a) $R_1(u;r,g_H)$ and $f_1(t;\gamma_F,f_a)$, or (b) $R_{1a}(u;r,1/g_H)$ and $f_{1a}(t;-\gamma_F,f_a)$. The discussed zero points A and B satisfy the relation $v_{Mx}(\Delta\varphi_{A,B})=0$, which implies that the stopping of the BF being under the action of the maximal deriving force was occurred. The $v_{Mx}-\Delta\varphi$ dependence shown by curve 2 is flattened; the dc drift induced by the low asymmetry ac force is less pronounced. The ratio of the maximal drift velocities taken at the extremes of both curves 2 and 1 approximately equals 0.4. Finally, curve 2 confirms the previous conclusion: small values of both the disbalance factor Δg_0 and the initial velocity of BF s_0 are required to achieve the discussed reversal with the low asymmetry ac force (see curve 2; zero points have disappeared).

In closing the discussion we summarize that the various types of the unforced dc drift of BFs are generated by the *symmetrical* rate functions if the driving force of zero mean is asymmetrically oscillating one. The characteristic features of the directed drift generated in the case of R_S-f_A symmetry are quite similar to those earlier obtained in the other case of the "opposite" symmetry labeled by R_A-f_S. The progressive (accelerated), regressive (decelerated) and reversal types of the unforced dc drift of BFs take place if the rate function is "unbalanced," namely, if Maxwellian construction of the rate function is not balanced. The parameters that govern the unforced drift are as follows: the asymmetry factor γ_F of the driving force, and both the slope parameter r and the balance factor g_H of the rate function. The decrease of the slope parameter enhances the driving effect. The dependencies of the drift velocity of BF versus the rest parameters, the asymmetry factor γ_F the balance factor g_H, generally speaking, are nonmonotone ones. The optimal asymmetry of the driving force, that induces the maximal driving effect, exists.

VI. CONCLUSIONS

The unidirectional drift of the bistable fronts that separate two stable uniform states of the bistable system under the action of the asymmetrically oscillating zero-mean force was considered within the "pseudolinear" model of the system. The unforced dc motion generated by the *symmetrical* (symmetrically shaped) rate functions $R_S(u)$ and the asymmetrically oscillating zero-mean driver $f_A(t)$ was examined. We found that the average characteristics generated in the case of

$R_S - f_A$ symmetry are quite similar to those given in the "opposite" case of $R_A - f_S$ symmetry, previously discussed in Ref. [8]. Different types of the unforced dc drift of the asymmetrically driven BFs were occurred. It was shown that the progressive (accelerated), regressive (decelerated), and reversal (regressive-progressive) types of the directed drift take place if the rate function is "unbalanced," namely, if Maxwellian construction of the rate function was not balanced. The "nonprogressive" (regressive, reversal, etc.) types of the unforced motion of BFs vanish if Maxwellian construction of the rate function is strictly balanced.

The governing parameters that significantly influence the directed drift of BFs are as follows: (a) the slope coefficients of the rate function, (b) the balance factor g_H, which indicates the disbalance of Maxwellian construction of the rate function from the strictly balanced situation, and (c) the asymmetry factor γ_F that indicates the "degree" of asymmetry of the driving force. The decrease of the slope parameter r enhances the driving effect; the shift of the drift velocity of the driven BF is more strongly pronounced at the lesser values of the parameter r. The dependence of the drift velocity versus the parameters g_H and γ_F, generally speaking, is nonmonotone. In particular, average velocity of the initially static (motionless) BF being under the action of the biharmonic ac force, which asymmetrically oscillates in time, increases if the asymmetry factor of the ac force increases. Differently, in the other case of the ac force, described by the periodic square-pulse function, the dependence of the drift velocity of BF versus the asymmetry factor γ_F is nonmonotone. The "optimal" asymmetry of the driving ac force that induces the maximal driving effect exists.

The particular case of the biharmonic driver consisting of a superposition of the fundamental mode and its even or odd superharmonics was considered. The biharmonic ac force with the odd harmonic mixing is rigorously symmetric, in any case of the "odd mixing." As a consequence, the average characteristics of BFs being under action of the biharmonic ac force described by a superposition of the odd harmonics are quite similar to those given in the case of the single-harmonic force. Differently, the biharmonic ac force described by a superposition of the fundamental mode and its even superharmonics is asymmetrically oscillating one. The "size" of the driving effect, the shift of the mean velocity of the asymmetrically driven BF being under action of the biharmonic ac force depends on the asymmetry factor γ_F, which is governed by the frequency, the relative phase and the relative amplitude of the superharmonic mode. The maximal driving effect of BF is achieved with the biharmonic force consisting of a superposition of the fundamental mode and its second superharmonic.

[1] P. Reimann, Phys. Rep. **361**, 57 (2002); D. Abbott, Chaos **11**, 526 (2001); A. M. Jayannavar, e-print cond-mat/0107079; F. Julicher, A. Adjari, and J. Prost, Rev. Mod. Phys. **69**, 1269 (1997); S. Denisov, S. Flach, A. A. Ovchinnikov, O. Yavtushenko, and Y. Zolotariyuk, Phys. Rev. E **66**, 041104 (2002). M. Borromeo, G. Constantin, and F. Marchesoni, *ibid.* **65**, 041110 (2002).

[2] D. Fleishman, A. F. Filipov, and M. Urbakh, Phys. Rev. E **69**, 011908 (2004).

[3] L. Schimansky-Geier and Ch. Zulicke Z. Phys. B: Condens. Matter **82**, 157 (1991); J. Armero, J. Casedemunt, L. Romirez-Piscina, and J. M. Sancho, Phys. Rev. E **58**, 5494 (1998); M. Santos and J. M. Sancho, *ibid.* **64**, 016129 (2001); Ch. R. Doering, C. Muller, and P. Smereka, Physica A **325**, 243 (2003); J. M. Sancho and A. Sancho, Eur. Phys. J. B **16**, 127 (2000); A. L. Sukstanskii and K. I. Primak, Phys. Rev. Lett. **75**, 3029 (1995); Y. S. Kivshar and A. Sanchez, *ibid.* **77**, 582 (1996); F. Marchesoni, *ibid.* **77**, 2364 (1996); N. Quintero, A. Sanchez, and F. G. Mertens, *ibid.* **84**, 871 (2000); M. Salerno and N. R. Quintero, Phys. Rev. E **66**, 041104 (2002).

[4] G. Costantini, F. Marchesoni, and M. Borromeo, Phys. Rev. E **65**, 051103 (2002); M. Salerno and N. R. Quintero, *ibid.* **65**, 025602(R) (2002); N. Quintero and A. Sanchez, Eur. Phys. J. B **6**, 133 (1998); M. Salerno and Y. Zolotaryuk, Phys. Rev. E **65**, 025602 (2002); S. Flach, Y. Zolotaryuk, A. E. Miriosh-nichenko, and M. V. Fistul, Phys. Rev. Lett. **88**, 184101 (2002).

[5] L. Morales-Molina, N. R. Quintero, F. G. Mertens, and A. Sanchez, Phys. Rev. Lett. **91**, 234102 (2003).

[6] A. Sanchez, L. Morales-Molina, F. G. Mertens, N. R. Quintero, J. Buceta, and K. Lindberg, e-print cond-mat/0403029; N. R. Quintero, B. Sanchez-Rey, and M. Salerno, e-print nlin.SI/0405023..

[7] F. G. Bass and R. Bakanas, Europhys. Lett. **53**, 444 (2001).

[8] R. Bakanas, Phys. Rev. E **69**, 016103 (2004).

[9] V. Mendez, J. Fort, and T. Pujol, J. Phys. A **36**, 3983 (2003); Sh.-I. Shima and Y. Kuramoto, Phys. Rev. E **69**, 036213 (2004); I. V. Barashenkov, S. Cross, and B. A. Malomed, *ibid.* **68**, 056605 (2003).

[10] R. Bakanas, Nonlinearity **16**, 313 (2003).